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The stability of Poiseuille flow in a pipe

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Numerical calculations show that the flow of viscous incompressible fluid in a circular pipe is stable to *small axisymmetric* disturbances at all Reynolds numbers. These calculations are linked with known asymptotic results.

1. Introduction

Osborne Reynolds studied the instability of Poiseuille flow of water in a pipe in his classic experiments at the end of the last century. He found that the laminar flow became unstable when $R = Wa/\nu$ was nearly 13,000, where W is the maximum velocity of parabolic basic flow at the middle of the circular pipe of radius a and ν is the kinematic viscosity of the fluid. Later experimenters found this critical value of the Reynolds number varied from about 2000, when the surface of the pipe is rough or the flow at the inlet irregular, to 40,000 or even more when the tube and the flow at the inlet are very smooth.

Sexl (1927*a*) began the theory of the instability of pipe flows. He found equations (1) of hydrodynamic stability for small axisymmetric disturbances of a basic flow in a pipe by the method of normal modes. Further theory on axisymmetric modes has been written by Corcos & Sellars (1959) and Gill (1965). Gill's paper is the most definitive, with a full account of the problem. A little inconclusive work on the harder problem of non-axisymmetric modes has been done by Sexl (1927*b*) and Lessen, Fox, Bhat & Liu (1964). It is now generally conjectured that Poiseuille flow is stable to infinitesimal disturbances but unstable to finite ones, on account of the experiments and of threads of theoretical evidence. The experimental instability has also been attributed to instability of the boundary layer inside the pipe at the inlet, upstream of the fully developed parabolic profile. Tatsumi (1952*b*) studied the instability of this boundary layer to only axisymmetric disturbances, finding a critical Reynolds number of nearly 10,000. It has been suggested also that instability may begin in the time taken to develop the parabolic profile. However, suitable experiments with gently converging inlet and a flow slowly built up are not susceptible to these mechanisms of instability and such experiments seem to be the rule.

The eigenvalue problem for small axisymmetric disturbances is stated in § 2 and it is shown that *unbounded* Poiseuille flow is stable. The two numerical methods we have used, integration of the stability equation with a 'shooting' method to match the two-point boundary conditions and an expansion of the eigenfunction as a series of known orthogonal functions, are described in §3. The stability characteristics of temporally and spatially damped disturbances of Poiseuille flow are presented in §4. The numerical results of the two methods agree and join up accurately with known asymptotic results, showing conclusively that all axisymmetric disturbances are stable at all Reynolds numbers.

2. The eigenvalue problem for axisymmetric disturbances

We shall work with dimensionless cylindrical polar co-ordinates (x, r, θ), choosing a as the unit of length and W of velocity. Then it can be shown (Sexl 1927*a*) that the stability of a pipe flow of incompressible viscous fluid with axisymmetric basic velocity $(U(r), 0, 0)$ to axisymmetric disturbances is governed by the equation

$$L^2\phi = i\alpha R\{(U-c)\,L\phi - r(U'/r)'\phi\}; \tag{1}$$

where the small radial velocity of the fluid

$$u_r = -r^{-1}\phi(r)\exp\{i\alpha(x-ct)\}, \tag{2}$$

α is the wave-number and $c = c_r + ic_i$ the complex wave velocity of the normal mode; dashes denote differentiations with respect to r; and the differential operator

$$L \equiv d^2/dr^2 - r^{-1}d/dr - \alpha^2. \tag{3}$$

The conditions of no slip on the pipe and of bounded velocity at the axis give

$$\phi, \phi' = 0 \quad (r = 0, 1). \tag{4}$$

Equation (1) and boundary conditions (4) specify an eigenvalue relation of the form

$$F(\alpha, c, R) = 0. \tag{5}$$

We have taken the view of temporally growing disturbances, for which real α is given and values of c are to be found as complex eigenvalues. Then there is instability of the flow if $\alpha c_i > 0$ for any mode. We may alternatively impose a real temporal frequency $\beta = \alpha c$ and seek complex eigenvalues $\alpha = \alpha_r + i\alpha_i$ as roots of relation (5). Then there is instability with a spatially growing mode if $\alpha_i < 0$ for any real β (Gaster 1962).

If we take the real part of

$$\int_0^1 (rR)^{-1}\,\phi^*(\text{equation (1)})\,dr,$$

integrate by parts and use boundary conditions (4), we readily find that

$$\alpha c_i\,(I_1^2 + \alpha^2 I_0^2) = \int_0^1 \tfrac{1}{2} i\alpha r^{-1} U'(\phi^{*\prime}\phi - \phi^*\phi')\,dr - R^{-1}(I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2) \tag{6}$$

for temporal modes, where an asterisk denotes a complex conjugate and the positive integrals

$$I_0^2 \equiv \int_0^1 r^{-1}|\phi|^2\,dr, \quad I_1^2 \equiv \int_0^1 r^{-1}|\phi'|^2\,dr, \quad I_2^2 \equiv \int_0^1 r\,|(r^{-1}\phi')'|^2\,dr. \tag{7}$$

This is in fact the power equation of the disturbance, the left-hand side being the rate of change of the kinetic energy of the disturbance, the integral the rate of transfer of energy from the basic flow to the disturbance by the Reynolds stress and the last term the rate of viscous dissipation of the disturbance per unit length of pipe. This result is essentially due to Synge (1938). He used variational methods on the power equation of plane parallel flows but obtained only weak sufficient conditions for stability, because the power equation contains no information of the singularity in the critical layer where $U = c$ which occurs in the limit as $\alpha R \to \infty$, a singularity intimately related to the mechanism of instability. So here we shall merely note that Synge has used a power equation to prove the stability of any pipe flow for sufficiently small R. In particular there is stability of a state of rest ($U \equiv 0$ or $R \equiv 0$).

However, we can gain more information by taking the real part of

$$\int_0^1 i\alpha(rR)^{-1}\phi^* L(\text{equation (1)})\, dr.$$

This gives the vorticity-square equation,

$$\alpha c_i (I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2) = \tfrac{1}{2}[r^{-1}|(L\phi)^2|']_0^1 + \tfrac{1}{2}i\alpha\int_0^1 r^{-1}\{r(U'/r)'\}'(\phi\phi^{*\prime} - \phi^*\phi')\, dr$$
$$- R^{-1}(I_3^2 + 3\alpha^2 I_2^2 + 3\alpha^4 I_1^2 + \alpha^6 I_0^2), \quad (8)$$

where
$$I_3^2 \equiv \int_0^1 r^{-1}|\{r(\phi'/r)'\}'|^2\, dr. \quad (9)$$

Note that (*a*) boundary condition (4) and the regularity of ϕ near $r = 0$ give $L\phi = 0$ at $r = 0$ and (*b*) equation (8) holds for unbounded flow if the upper limit one is replaced by infinity. Now suppose that $U = A + Br^2$ $(0 \leqslant r < \infty)$ for some constants A, B. Then condition (4) at infinity implies that the solution ϕ of (1) tends to zero exponentially and therefore that $L\phi \to 0$ as $r \to \infty$. In this case it follows from (8) that $\alpha c_i < 0$ and therefore that unbounded parabolic flows are stable to axisymmetric disturbances. (It may be analogously shown that unbounded plane Couette and plane Poiseuille flows are stable.)

3. Numerical procedures

We have used two numerical procedures. For the first, the solution of the stability equation (1) is expanded as a power series in r near the channel centre. From a small but finite value of r the solution is continued by Runge–Kutta integration to the wall. Various tests were made to optimize the size and number of the steps of integration, smaller steps being necessary where the solution varied more rapidly. To solve the two-point boundary-value problem a two-tier shooting technique was used. This was necessary because for large αR the exponents of the two unwanted exponentially growing solutions have different orders of magnitude. No filter was necessary because the computer had efficient multi-precision facilities and the results join accurately with asymptotic results.

For the second numerical procedure, we considered the expansion of the solution

$$\phi = \sum_{n=1}^{\infty} a_n \phi_n,$$

where $\{\phi_n\}$ is some known complete set of functions satisfying the boundary conditions and $\{a_n\}$ is a sequence to be determined from (1). This method has been applied to the stability of plane parallel flows by Dolph & Lewis (1958), Gallagher & Mercer (1962) and Grosch & Salwen (1968). To get rapid convergence $\{\phi_n\}$ should be close to the actual sequence of eigenfunctions of the system (1), (4). However, this is unpractical and it seems best to define ϕ_n as the nth eigenfunction with value λ_n ($< \lambda_{n+1}$) satisfying boundary conditions (4) and the equation

$$L^2\phi = -\lambda\phi. \tag{10}$$

We can identify λ with $i\alpha Rc$ and $\{\phi_n\}$ with the eigenfunctions of the stability problem (1), (4) when $U \equiv 0$ or $R \to 0$. It can be shown (Sexl 1927*a*) that

$$\phi_n = \mu_n^{-\frac{1}{2}} r\{I_1(\alpha) J_1(u_n r) - J_1(u_n) I_1(\alpha r)\}; \tag{11}$$

where μ_n is any constant, $\lambda_n = u_n^2 + \alpha^2$, u_n ($< u_{n+1}$) is the real nth root of the transcendental equation

$$uJ_1'(u)/J_1(u) = -\beta \equiv \alpha I_1'(\alpha)/I_1(\alpha), \tag{12}$$

J_1 is the Bessel function and I_1 the modified Bessel function of order one. It can be seen that $-\infty < \beta < -1$ for each $\alpha > 0$, and hence that

$$j_{1,n} < u_n < j_{1,n+1} \quad (n = 1, 2, 3, \ldots),$$

where $j_{1,n}$, $j'_{1,n}$ are the nth zeros of $J_1(u)$, $J_1'(u)$ respectively. The eigenfunctions $\{\phi_n\}$ have the orthogonality property that

$$\int_0^1 -r^{-1}\phi_m L\phi_n \, dr = \int_0^1 r^{-1}(\phi_m' \phi_n' + \alpha^2 \phi_m \phi_n)\, dr = \delta_{mn}, \tag{13}$$

when we have normalized by choosing

$$\begin{aligned}\mu_n &= \lambda_n I_1(\alpha) \int_0^1 r\{I_1(\alpha) J_1(u_n r) - J_1(u_n) I_1(\alpha r)\} J_1(u_n r)\, dr \\ &= \lambda_n(\beta^2 + u_n^2 - 1) I_1^2(\alpha) J_1^2(u_n)/2u_n^2.\end{aligned} \tag{14}$$

Substitution of the series for ϕ into (1) gives

$$\sum_{n=1}^{\infty} \{(U-c) L\phi_n - r(U'/r)'\phi_n - i\lambda_n(\alpha R)^{-1} L\phi_n\} a_n = 0. \tag{15}$$

$\int_0^1 r^{-1}\phi_m$ (equation (15)) dr gives the infinite set of algebraic equations

$$\sum_{n=1}^{\infty} \{(c + i\lambda_n/\alpha R)\delta_{mn} + B_{mn}\} a_n = 0, \tag{16}$$

where
$$B_{mn} \equiv \int_0^1 \phi_m\{r^{-1} U L\phi_n - (U'/r)'\phi_n\}\, dr. \tag{17}$$

In our numerical work the infinite system (16) is truncated, the series being summed to N instead of infinity, with $N = 27$. Then the first N roots c were found by procedures COMHES and COMLR of Martin & Wilkinson (1968*a*, *b*) as the eigenvalues of the $N \times N$ matrix $\mathrm{diag}(-i\lambda_n/\alpha R) - B_{mn}$. Convergence is discussed by Dolph & Lewis (1958) for the analogous problem of plane parallel flows. The truncation error is difficult to assess in practice, but increases with αR because the terms $\lambda_n/\alpha R$ diminish relative to B_{mn} as αR increases, yet $B_{nn} \to -\frac{2}{3}$, not zero, as $n \to \infty$.

Perhaps the most convincing evidence of the accuracy of our two numerical procedures is their agreement on the eigenvalues c (the magnitude of the difference of their two results being typically 10^{-4} for $\alpha R < 10{,}000$) and their agreement with asymptotic results. The first method of direct integration is by and large superior, being quicker and more flexible. In particular it gives eigenfunctions and spatially growing modes easily. The method of function expansion seems better for finding all the first N eigenvalues of the temporally growing modes rather than the first one or two and for showing clearly which mode is which.

4. Stability characteristics of Poiseuille flow

First we shall describe the characteristics of temporally damped disturbances of Poiseuille flow,

$$U = 1 - r^2 \quad (0 \leqslant r \leqslant 1). \tag{18}$$

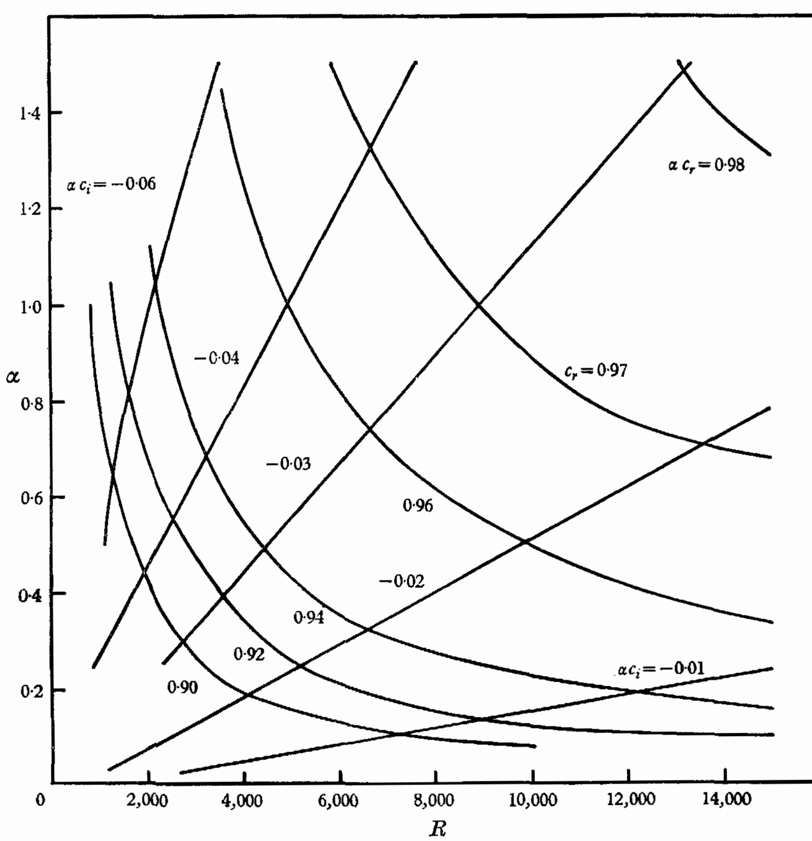

FIGURE 1. Curves of constant αc_i and c_r in the (α, R)-plane for the least stable temporal mode ($m = 1$ or $n = 1$).

Each mode was found to be stable for all real α, R. Figure 1 gives curves of constant phase-velocity c_r and logarithmic growth rate $\beta_i = \alpha c_i$ of the least stable mode. It can be seen that the curves of constant β_i are nearly straight lines. This is a centre mode for which $c_r \to 1$ as $\alpha R \to \infty$. Pekeris found these modes with eigenvalue

$$c = 1 + 4m e^{-\frac{3}{4}\pi i} (\alpha R)^{-\frac{1}{2}} + o(\alpha R)^{-\frac{1}{2}} \quad \text{as} \quad \alpha R \to \infty \quad (m = 1, 2, 3, \ldots) \tag{19}$$

(cf. Gill 1965, p. 156). Our results agree with this formula well for $m = 1$ and

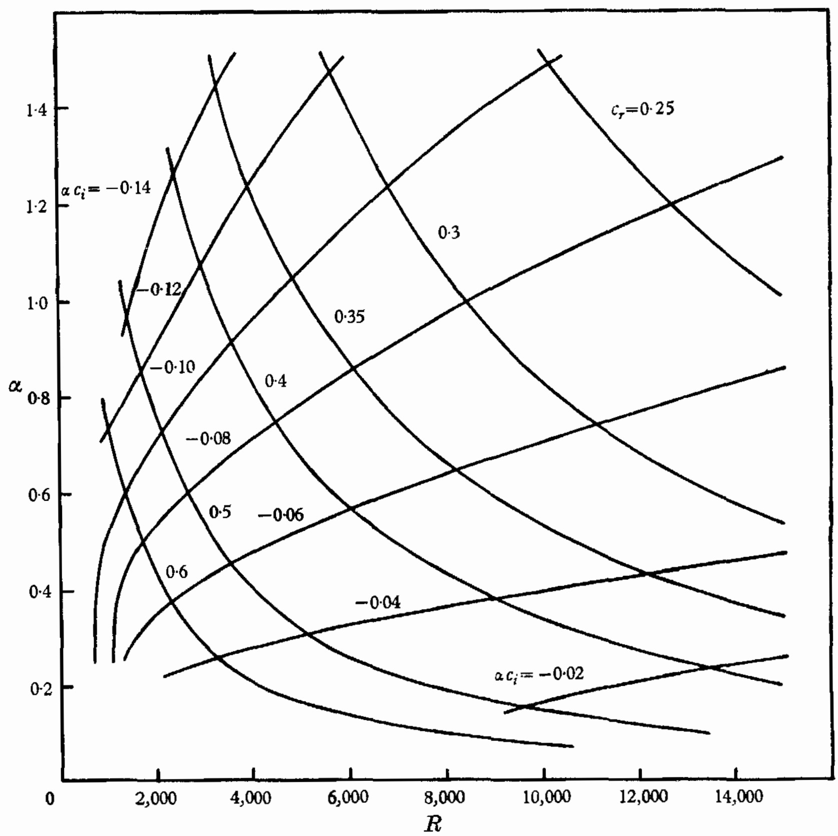

FIGURE 2. Curves of constant αc_i and c_r in the (α, R)-plane for the least stable temporal wall mode ($q = 1$ or $n = 2$).

αR more than about 5000. Corcos & Sellars (cf. Gill 1965, p. 161) found also wall modes for which

$$c = -2^{\frac{2}{3}} e^{-\frac{1}{6}\pi i} z_{\pm q} (\alpha R)^{-\frac{1}{3}} + o(\alpha R)^{-\frac{1}{3}} \quad \text{as} \quad \alpha R \to \infty, \tag{20}$$

where z_q is the qth root with positive imaginary part of

$$\int_{-z_q}^{\infty} \mathrm{Ai}(z)\, dz = 0 \quad (q = 1, 2, 3, \ldots) \tag{21}$$

and $z_{-q} = z_q^*$. Figure 2 gives curves of constant β_i, c_r for the least stable wall mode ($q = 1$). From a temporal point of view this mode is the second least stable for small αR, the third for most points of the (α, R)-plane in figure 2, but more stable than the centre modes as $\alpha R \to \infty$.

Figure 3(*a*) gives c_r against c_i for $\alpha = 1, R = 10{,}000$ for 27 modes c_n. The similarity with Corcos & Sellars's (1959, figure 2) asymptotic results is striking, although $(\alpha R)^{-\frac{1}{3}}$ is not very small when $\alpha R = 10^4$ and c_r less so. Figure 3(*b*) and other results

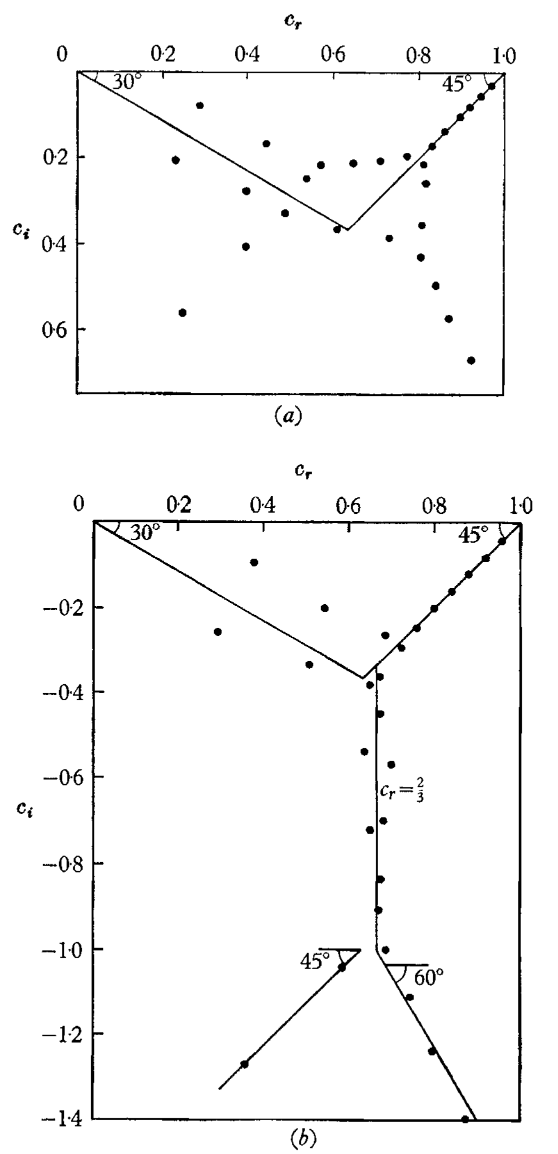

FIGURE 3. (*a*) Values of c for $\alpha = 1$, $R = 10{,}000$. (*b*) Values of c for $\alpha = 1$, $R = 5000$.

which we have obtained, but are not presented here in detail, suggest the following pattern of modes in the complex c-plane: the infinity of points c_n lie near either the line $c_r = \frac{2}{3}$ or pairs of lines radiating from points with $c_r = \frac{2}{3}$ and making angles of 45° or −60° and −30° or 45° respectively with the positive c_r-axis, and as αR increases the points c_n move along these lines towards the c_r-axis. For αR

less than about 100, our results agree well with equation (12), etc., giving

$$\alpha c R \sim -i\lambda_n(\alpha).$$

Next, we give our results for spatially damped disturbances, which have their frequency fixed and real and which decay downstream with a logarithmic damping rate α_i. The temporal wall mode $q = 1$ has a high value of c_r and thus

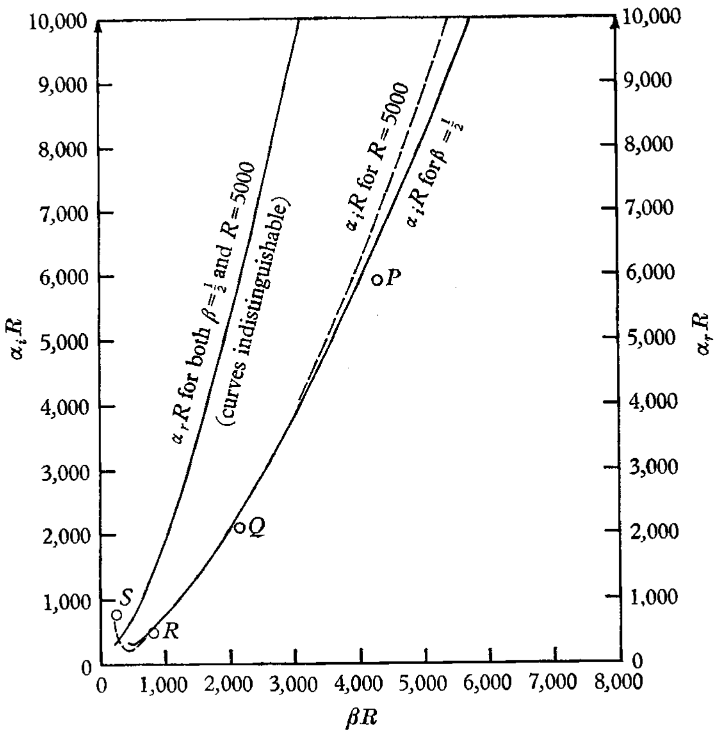

FIGURE 4. Curves $\beta = \frac{1}{2}$ and $R = 5000$ in the $(\alpha_r, \beta R)$- and $(\alpha_i, \beta R)$-planes. Results correspond to the mode $q = 1$.

Gaster's transformation may be used on the temporal results to obtain some approximate spatial solutions. These were used as initial data for the shooting-technique numerical method. The results for this wall mode are summarized in figure 4, which gives curves of β = constant and R = constant in the $(\alpha_i R, \beta R)$- and $(\alpha_r R, \beta R)$-planes. We have taken $\beta = \frac{1}{2}$ and $R = 5000$ as typical values. This figure may be compared with figure 5(*b*) of Gill (1965) with regard to the $q = 1$ mode. For comparison with Gill's asymptotic theory, the point P on figure 4 is given by equation (5.8) of Gill with $\beta R = 4280$. However, as (5.8) is an approximation for small β, P must lie on the curve $\beta = 0$ or its limiting position in figure 4. Points Q, R, S correspond to $\beta R = 2140$, 800, 236 respectively with excellent agreement.

The mode $q = 1$ is the least damped wall mode, but it is more highly damped than several of the l- and m-modes defined by Gill. We now attend to these less damped modes, which for small values of βR (the l-modes) cover the whole of the pipe and for larger values of βR (the m-modes) are concentrated near the centre. In equation (6.22) of his paper Gill gave limiting values of $(\alpha_i R)^{\frac{1}{2}}$ as $\beta R \to 0$ for the first three l-modes, using the asymptotic approximation for large l. He gave values for $c_r = \beta/\alpha_r$ of 0·9161, 0·8510, 0·8137 and we found these values to be 0·8264, 0·7615 and 0·7173 respectively on solving the eigenvalue problem numerically.

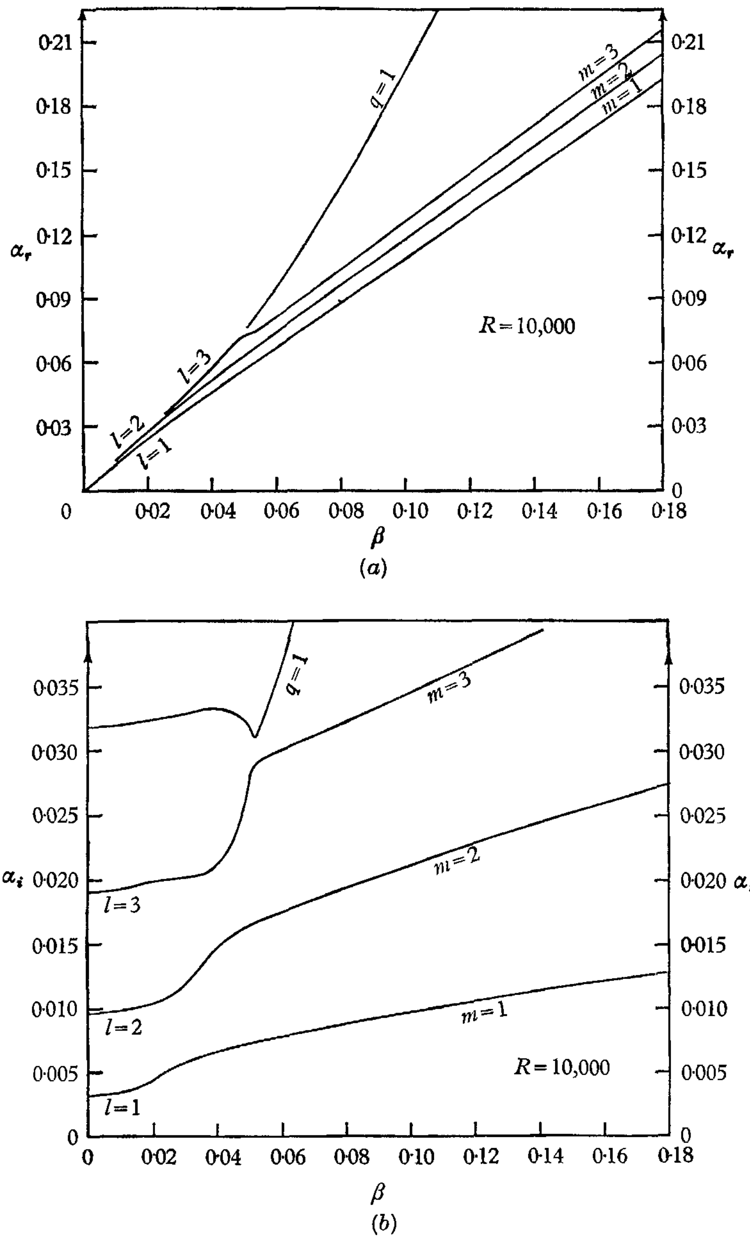

FIGURE 5. (*a*) Curves $\alpha_r(\beta)$ for R = 10,000 and modes m = 1, 2, 3; q = 1. (*b*) Curves $\alpha_i(\beta)$ for R = 10,000 and modes m = 1, 2, 3; q = 1.

Figure 5(a) gives the variation of α_r and figure 5(b) the variation of α_i with small β for each of the modes l or $m = 1, 2, 3$ and $q = 1$ when $R = 10{,}000$. Our figure 5(a) corresponds closely with Gill's figure 5(b). The curves of our figure 5(b) pass through the exact values given by Gill's equation (6.22) for $\beta = 0$. In all, our calculations, presented or not presented here, support Gill's interpretation and interpolation of the asymptotic results, even agreeing on the peculiar 'kink' in the graph of α_i against βR for the centre modes.

5. Conclusions

In view of the long controversy over the instability of Poiseuille flow, with 'proofs' and objections to 'proofs', it is reassuring that our numerical results join so well with asymptotic results, even for values of $(\alpha R)^{\frac{1}{3}}$ that are not very large. It emerges clearly now that the flow is stable to all axisymmetric small disturbances, that there is an infinite number of such modes and that Gill's (1965) interpretation of the asymptotic results and of the relation between the modes is confirmed.

Nevertheless, non-axisymmetric infinitesimal disturbances and finite amplitude disturbances, which are the most likely causes of instability, have, as yet, received very little theoretical or computational treatment. The value of the Landau constant for finite amplitude axisymmetric disturbances is being calculated.

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